BOUNDARY LAYER AT A PLATE WITH AN ARBITRARY INJECTION MODE

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A sufficiently general solution is obtained, applicable to a boundary layer with any injection mode (up to a certain limit).

1. We consider a horizontal plate in a laminar air stream. A gas is injected into the boundary layer through the plate. The air is considered a homogeneous diatomic gas. In this way, there exists a binary mixture in the boundary layer. Any dissociation or chemical reaction is disregarded, and thermal diffusion is assumed negligible.

The solutions to this problem found in the technical literature [1-3] are usually given in the self-adjoint form and correspond to a specific mode of injection through the surface.

The equations for a steady laminar boundary layer of a binary mixture at a plate can, after a Dorodnitsyn–Lis transformation [4, 5], be written as

the equation of motion

$$lf_{\eta\eta\eta} + ff_{\eta\eta} = 2\xi\,(f_{\eta\xi}f_\eta - f_{\eta\eta}f_\xi),$$

the equation of diffusion

$$\frac{l\mathrm{Le}}{\mathrm{Pr}} S_{\eta\eta} + fS_\eta = 2\xi\,(S_\xi f_\eta - S_\eta f_\xi), \tag{1}$$

the energy equation

$$\frac{l}{\mathrm{Pr}}\,\theta_{\eta\eta} + f\theta_\eta + l\,(k-1)\,\mathrm{M}_\infty^2 f_\eta^2 + \frac{c_{p_2} - c_{p_1}}{\bar{c}_p}\,\frac{l\,\mathrm{Le}}{\mathrm{Pr}}\,S_\eta\theta_\eta = 2\xi\,(f_\eta\theta_\xi - f_\xi\theta_\eta).$$

The continuity equation becomes an identity by introduction of the flow function, with the following designations:

$$\eta = \frac{u_\infty}{\sqrt{2\xi}}\int_0^y \rho\,dy;\quad \xi = \int_0^x \rho_w\mu_w u_\infty dx;$$

$$u = \frac{1}{2}u_\infty\frac{\partial f}{\partial\eta};\quad \psi = \frac{1}{2}\sqrt{2\xi}\,f(\xi,\ \eta);$$

$$\bar{c}_p = \sum_{i=1}^{2} c_i c_{pi}\,;\ \mathrm{Pr} = \frac{\mu\bar{c}_p}{\lambda};\ \mathrm{Le} = \frac{\rho D_{12}\bar{c}_p}{\lambda};\quad l = \frac{\rho\mu}{\rho_w\mu_w} \sim 1;\quad \theta = \frac{T}{T_\infty};$$

$$k = \frac{c_p}{c_v};\ \mathrm{M}_\infty = \frac{u_\infty}{a_\infty}.$$

Subscripts 1 and 2 refer to pure coolant and to pure air, respectively.

The plate temperature is held constant. The physical properties which appear in the equations are assumed constant.

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The boundary conditions are:

$$\text{for } \eta = 0:\ f_\eta(0) = 0;\ \theta = \theta_w = \text{const};\ (f + 2\xi f_\xi)_w = (f + 2\xi f_\xi)_w S_w + \frac{2\mathrm{Le}}{\mathrm{Pr}} \frac{\partial S_w}{\partial \eta};\ f(\xi,\ 0) = -\frac{\sqrt{2}}{\rho_w \mu_w u_\infty \sqrt{\xi}} \int_0^\xi F d\xi \equiv \Phi(\xi); \tag{2}$$

$$\text{for } \eta \to \infty:\ f_\eta \to 2;\ \theta \to 1;\ S \to 0.$$

It is to be noted that stipulating the injection mode as $\rho_W v_W \sim 1/\sqrt{x}$ will render the problem a self-adjoint one and will transform the system of boundary-layer equations into a system of ordinary differential equations [6].

2. System (1) with the boundary conditions (2) is solved by the method shown by Shkadov in [7-9], i.e., by series in variables $\Phi(\xi)$, $\xi\Phi'(\xi)$, $\xi^2\Phi''(\xi)$, ... The coefficients of these series are functions only of η and they satisfy the ordinary differential equations. Inasmuch as one can solve separately the equation of motion, then the equation of diffusion, and finally the energy equation, we will carefully analyze the equation of motion (1) with the thought that the solution of the other two equations is analogous.

Substituting $f(\xi, \eta) = \Phi(\xi) + \bar{f}(\xi, \eta)$ in the equation of motion yields

$$\bar{f}_{\eta\eta\eta} + (\Phi + \bar{f} + 2\xi \bar{f}_\xi + 2\xi\Phi')\bar{f}_{\eta\eta} - 2\xi \bar{f}_{\xi\eta} \bar{f}_\eta = 0, \tag{3}$$

and the boundary conditions

$$\text{for } \eta = 0:\ \bar{f} = 0;\ \bar{f}_\eta = 0;$$

$$\text{for } \eta \to \infty:\ \bar{f}_\eta \to 2.$$

The dash above f will henceforth be omitted. We seek a solution to (3) in the form of a series

$$f = f_{00} + \alpha_0 f_{01} + \alpha_0^2 f_{02} + \alpha_1 (f_{10} + \alpha_0 f_{11} + \alpha_0^2 f_{12}) + \alpha_1^2 (f_{20} + \alpha_0 f_{21}) + \alpha_2 (f_{30} + \alpha_0 f_{31}) + \alpha_1^3 (f_{40} + \alpha_0 f_{41}) + \alpha_1\alpha_2 (f_{50} + \alpha_0 f_{51}) + \alpha_3 (f_{60} + \alpha_0 f_{61}) + \dots, \tag{4}$$

where

$$\alpha_0 = \Phi(\xi);\ \alpha_0^2 = \Phi^2;\ \alpha_1 = \xi \frac{d\Phi}{d\xi} = \xi\Phi';\ \alpha_2 = \xi^2\Phi'',\ \dots,$$

and f_{ij} are functions of η only.

Inserting (4) into (3), then collecting terms with the same combinations of α_j^i and equating them to zero, we obtain the ordinary differential equations

$$\begin{gathered} \frac{d^3 f_{00}}{d\eta^3} + f_{00} \frac{d^2 f_{00}}{d\eta^2} = 0; \\ f_{01}''' + f_{00} f_{01}'' + f_{00}'' f_{01} = -f_{00}''; \\ f_{02}''' + f_{00} f_{02}'' + f_{00}'' f_{02} = -(1 + f_{01}) f_{01}''; \\ f_{10}''' + f_{00} f_{10}'' - 2 f_{00}' f_{10}' + 3 f_{00}'' f_{10} = 2 f_{01}' f_{00}' - 2(1 + f_{01}) f_{00}'', \end{gathered} \tag{5}$$

and in the general form

$$f_{ij}''' + f_{00} f_{ij}'' - k_1 f_{00}' f_{ij}' + k_2 f_{00}'' f_{ij} = A(\eta).$$

The boundary conditions are for f_{00}:

$$\eta = 0;\ f_{00} = 0;\ f_{00}' = 0;$$

$$\eta \to \infty;\ f_{00}' \to 2,$$

and for the other f_{ij}:

$$\eta = 0;\ f_{ij} = f_{ij}' = 0;$$

$$\eta \to \infty;\ f_{ij}' \to 0.$$

Analogously, for the equation of diffusion we have

$$S_{00} = 0;$$

$$\frac{\mathrm{Le}}{\mathrm{Pr}} S_{01}'' + f_{00} S_{01}' = 0;$$

$$\frac{\mathrm{Le}}{\mathrm{Pr}} S_{02}'' + f_{00} S_{02}' = -f_{01} S_{01}';$$

$$\frac{\mathrm{Le}}{\mathrm{Pr}} S_{10}'' + f_{00} S_{10}' - 2 f_{00}' S_{10} = 2 f_{00}' S_{01},$$

and in the general form

$$\frac{\mathrm{Le}}{\mathrm{Pr}} S_{ij}'' + f_{00} S_{ij}' - k_3 f_{00}' S_{ij} = B(\eta).$$

For the energy equation we have

$$\frac{1}{\mathrm{Pr}} \theta_{00}'' + f_{00}\theta_{00}' = -(k-1) \mathrm{M}_\infty^2 f_{00}''^2;$$

$$\frac{1}{\mathrm{Pr}} \theta_{01}'' + f_{00}\theta_{01}' = -2(k-1)\mathrm{M}_\infty^2 f_{00}' f_{01}' - \theta_{00}' \left(f_{01} + \frac{c_{p2} - c_{p1}}{\bar{c}_p} \frac{\mathrm{Le}}{\mathrm{Pr}} S_{01}' \right);$$

$$\frac{1}{\mathrm{Pr}} \theta_{02}'' + f_{00}\theta_{02}' = -(k-1)(f_{01}'^2 + 2 f_{00}' f_{02}') - \theta_{00}' f_{02} - \theta_{01}' f_{01} - \frac{(c_{p2} - c_{p1})}{\bar{c}_p} (S_{02}' \theta_{00}' + \theta_{01}' S_{01}'),$$

and in the general form

$$\frac{1}{\mathrm{Pr}} \theta_{ij}'' + f_{00}\theta_{ij}' - k_4 f_{00}' \theta_{ij} = C(\eta).$$

The boundary conditions are

$$\text{at } \eta = 0\text{: } S_{ij}' = L(\eta);\ \theta_{00} = \theta_w;\ \theta_{ij} = 0;$$

$$\text{at } \eta \to \infty\text{: } S_{ij}' \to 0;\ \theta_{00} = 1;\ \theta_{ij} \to 0.$$

We note that the first equation in (5) for f_{00} with the corresponding boundary conditions represents the well-known Blasius problem [10].

3. Equations (5) were integrated numerically by conversion to a system of difference equations, which was then solved by the elimination method. The elimination formulas for the third-order equation of motion are shown here in detail. With the aid of the central differences [11], we obtain the following elimination formulas:

$$f_i = P_i f_{i+2} + Q_i f_{i+1} + R_i, \tag{6}$$

where

$$P_i = -\frac{1}{2\left[\left(1 + h f_{00i} + \frac{1}{2} k_1 h^2 f_{00i}' - \frac{1}{2} Q_{i-2}\right) Q_{i-1} + k_2 h^3 f_{00i}'' - 2h f_{00i} - \frac{1}{2} P_{i-2}\right]} \equiv -\frac{1}{2F_1};$$

$$Q_i = \frac{\left(1 + h f_{00i} + \frac{1}{2} k_1 h^2 f_{00i}' - \frac{1}{2} Q_{i-2}\right) P_{i-1} - \left(1 - h f_{00i} + \frac{1}{2} k_1 h^2 f_{00i}'\right)}{F_1};$$

$$R_i = \frac{\frac{1}{2} R_{i-2} - \left(1 + h f_{00i} + \frac{1}{2} k_1 h^2 f_{00i}' - \frac{1}{2} Q_{i-2}\right) R_{i-1} + h^3 A_i}{F_1}$$

$$(i = 2,\ 3,\ \ldots,\ n-2);$$

$$P_0 = Q_0 = R_0 = 0;\ P_1 = 0;\ Q_1 = \frac{1}{4};\ R_1 = 0.$$

With the aid of the eliminating relations at points $n-2$, $n-3$, $n-4$, the difference equation at point $n-1$, and the boundary condition on the right-hand side, we find f_n and f_{n-1}. Then, by reverse elimination, from (6) we find f_{n-2}, $f_{n-3}, \ldots, f_1$.

Computer programs have been set up for solving Eqs. (5), and calculations were made with Le = 1.4, Pr = 0.725, Ma_∞ = 3, k = 1.4, T_W = 500°K, T_∞ = 273°K, and c_{p2}, c_{p1} taken from [12]. The integration step

TABLE 1. Values of the Derivatives of f, θ, and S

$f''_{00}(0)$	0,33206	$\theta'_{00}(0)$	−0,11772	$S'_{00}(0)$	0
$f''_{01}(0)$	0,71861	$\theta'_{01}(0)$	−0,10772	$S'_{01}(0)$	−0,25892
$f''_{02}(0)$	−0,44048	$\theta'_{02}(0)$	−0,10770	$S'_{02}(0)$	−0,19943
$f''_{10}(0)$	0,00042	$\theta'_{10}(0)$	0,80177	$S'_{10}(0)$	−0,51785
$f''_{11}(0)$	0,00041	$\theta'_{11}(0)$	0,72224	$S'_{11}(0)$	−0,51447
$f''_{12}(0)$	0,00440	$\theta'_{12}(0)$	0,62072	$S'_{12}(0)$	−0,51192
$f''_{20}(0)$	−0,01260	$\theta'_{20}(0)$	0,73810	$S'_{20}(0)$	−0,25892
$f''_{21}(0)$	−0,04115	$\theta'_{21}(0)$	0,40360	$S'_{21}(0)$	−0,23878
$f''_{30}(0)$	−0,00804	$\theta'_{30}(0)$	0,61002	$S'_{30}(0)$	−0,66556
$f''_{31}(0)$	0,00777	$\theta'_{31}(0)$	0,45677	$S'_{31}(0)$	−0,67517
$f''_{40}(0)$	−0,67819	$\theta'_{40}(0)$	0,36460	$S'_{40}(0)$	−0,50792
$f''_{60}(0)$	0,35302	$\theta'_{60}(0)$	0,61030	$S'_{60}(0)$	−0,09527
$f''_{61}(0)$	0,03392	$\theta'_{61}(0)$	0,45746	$S'_{61}(0)$	−0,09525

TABLE 2. Values of Skin Friction, Thermal Flux, and Diffusion Current

c_1	0	0,04	0,06	0,1
τ/τ_0	1	0,770	0,515	0,232
q/q_0	1	0,915	0,845	0,807
q_g	0	0,0239	0,0438	0,0597

was h = 0.04. The integration interval was η = 7.5, where f_{ij} already closely approached the asymptote. Calculated were also the skin friction as well as the thermal fluxes and the diffusion currents:

$$\tau = \mu_w \left(\frac{\partial u}{\partial y}\right)_w \quad \text{or} \quad \sqrt{2}\, c_f \sqrt{\mathrm{Re}_x} = f''_{ij}(0),$$

$$q = -\lambda_w \left(\frac{\partial T}{\partial y}\right)_w \quad \text{or} \quad \sqrt{2}\,\mathrm{Nu}/\sqrt{\mathrm{Re}_x} = -\theta'_{ij}(0), \tag{7}$$

$$q_g = D_w \left(\frac{\partial c}{\partial y}\right)_w \quad \text{or} \quad \sqrt{2}\,\mathrm{Nu}_g/\sqrt{\mathrm{Re}_x} = S'_{ij}(0),$$

with $f''_{ij}(0)$, $\theta'_{ij}(0)$, and $S'_{ij}(0)$ given in Table 1.

4. A more accurate quantitative comparison between the self-adjoint case and an arbitrary case will be made and advantages of an arbitrary injection mode will be demonstrated on specific examples.

First we consider the well-known self-adjoint case:

$$\rho_w v_w = F(x) = \frac{\beta}{\sqrt{\xi}}. \tag{8}$$

From the boundary condition (2)

$$f(0,\ \xi) = -\frac{\sqrt{2}\,\xi^{-1/2}}{\rho_w \mu_w u_\infty} \int_0^{\xi} F d\xi \equiv \Phi(\xi)$$

we obtain

$$\alpha_0 \equiv \Phi = -\frac{2\sqrt{2}\beta}{\rho_w \mu_w u_\infty} \equiv c = \text{const};\ \alpha_1 = \alpha_2 = \ldots = 0. \tag{9}$$

From (8) and (9), after a transformation, we obtain

$$\frac{v_w}{u_\infty} = \frac{c_1}{\sqrt{\mathrm{Re}_x}},$$

where $c_1 = c/2\sqrt{2}$ characterizes the rate of gas injection through the surface.

Relative values of skin friction, thermal flux, and diffusion current for various values of c_1 are given in Table 2, where τ_0 and q_0 are the respective values without injection.

According to Table 2, τ and q decrease with increasing injection rate, which agrees with theory.

b) For a nonself-adjoint case we selected the following injection mode:

$$\rho_w v_w = \frac{\rho_w \mu_w u_\infty}{2\sqrt{2}} \beta \sqrt{\alpha}\, \frac{(3+\sqrt{\bar{\xi}})\sqrt{\bar{\xi}}}{(1+\bar{\xi})^2},$$

with $\bar{\xi} = \xi/\alpha$. The dash above ξ will henceforth be omitted. Then

$$\alpha_0 \equiv \Phi(\xi) = -\beta\frac{\xi}{1+\xi},\ \alpha_1 \equiv \xi\Phi' = -\alpha\beta\,\frac{\xi}{(1+\xi)^2},$$

$$\alpha_2 \equiv \xi^2\Phi'' = 2\alpha^2\beta\frac{\xi^2}{(1+\xi)^3},\ \alpha_3 \equiv \xi^3\Phi''' = -6\alpha^3\beta\,\frac{\xi^3}{(1+\xi)^4}. \tag{10}$$

Letting $\alpha = 1$, inserting (10) into expressions (7) for τ, q, and q_g, we obtain the desired values. A comparison between both cases shows that

$$\xi \to 0 \text{ when } \rho_w v_w \sim \frac{1}{\sqrt{\xi}} \to \infty, \qquad \text{for } \rho_w v_w \sim \frac{(3+\sqrt{\xi})\sqrt{\xi}}{(1+\xi)^2} \to 0,$$

i.e., $\rho_W v_W$ is a bounded quantity, which has some practical significance. In both cases a higher rate of gas injection through the surface results in a lower skin friction and thermal flux, which agrees with theory, the curves of skin friction and of thermal flux for $\rho_W v_W \sim 1/\sqrt{\xi}$ being asymptotes of the respective curves for $\rho_W v_W \sim (3 + \sqrt{\xi})\sqrt{\xi}/(1 + \xi)^2$.

NOTATION

ρ is the density of gas;
μ is the dynamic viscosity of gas;
η, ξ are the dimensionless coordinates;
f is the dimensionless gas velocity;
S is the dimensionless gas concentration;
θ is the dimensionless gas temperature;
λ is the thermal conductivity;
D_{12} is the diffusivity.

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